Suppose that we have an object that consists of a continuum of particles, and ρ denote the density of mass of the object. Let B denote the extent of the body itself or the region of the 3 dimensional space occupied by the body.

$$\int_B \rho(x, y, z)\, dx\, dy\, dz = m$$

density.

mass

The kinetic energy of the object is given by

$$K = \frac{1}{2} \int_B V^T(x, y, z)\, V(x, y, z)\, \rho(x, y, z)\, dx\, dy\, dz$$

$$= \frac{1}{2} \int_B V^T(x, y, z)\, V(x, y, z)\, dm.$$

infinitismal mass of the particle located at (x, y, z).

The center of mass of the object has the coordinates (x_c, y_c, z_c) defined by

$$x_c = \frac{1}{m} \int_B x\, dm$$

$$y_c = \frac{1}{m} \int_B y\, dm$$

$$z_c = \int_B z\,dm.$$

In a more compact form, we have

$$r_c = \frac{1}{m}\int_B r\,dm$$

$$\begin{bmatrix} x_c \\ y_c \\ z_c \end{bmatrix} \qquad \begin{bmatrix} x \\ y \\ z \end{bmatrix}$$

We have

$$\int_B (r_c - r)\,dm = \emptyset$$

We attach a coordinate system rigidly to the center of mass. As the body moves the velocities of the object is given by

$$V = V_c + \underbrace{\omega \overset{\text{cross product.}}{\times} r}_{\text{can be written as } S(\omega)r}$$

Let R denote the rotation matrix that transforms from the moving frame to the inertial frame.

The velocity of the particle located at r, expressed w.r.t. the moving frame is given by

$$R^T(V_c + \omega \times r) = R^T V_c + (R^T\omega) \times (R^T r)$$

The kinetic energy is then:

$$K = \frac{1}{2}\int_B (V_c + S(\omega) r)^T (V_c + S(\omega) r)\, dm$$

$$= K_1 + K_2 + K_3 + K_4$$

$$K_1 = \frac{1}{2}\int_B V_c^T V_c \, dm = \frac{1}{2} m V_c^T V_c$$

This is called the translation part of the kinetic energy.

$$K_2 = \frac{1}{2}\int_B V_c^T S(\omega) r \, dm = \frac{1}{2} V_c^T S(\omega) \int_B r \, dm$$

Since

$$\int_B r \, dm = 0$$

Because the center of mass is at the origin of the coordinate system

$$\therefore \quad K_2 = 0$$

Likewise

$$K_3 = 0$$

$$K_4 = \frac{1}{2}\int_B r^T S^T(\omega) S(\omega)\, r \, dm$$

$$= \frac{1}{2}\, \mathrm{Tr}\left(S(\omega) J S^T(\omega) \right)$$

trace: the sum of the diagonal elements.

$$J = \int_B r r^T \, dm$$

$$J = \begin{bmatrix} \int_B x^2 dm & \int_B xy\, dm & \int_B xz\, dm \\ \int_B xy\, dm & \int_B y^2 dm & \int_B yz\, dm \\ \int_B xz\, dm & \int_B xy\, dm & \int_B z^2 dm \end{bmatrix}$$

$$S(\omega) = \begin{bmatrix} 0 & -\omega_3 & \omega_2 \\ \omega_3 & 0 & -\omega_1 \\ -\omega_2 & \omega_1 & 0 \end{bmatrix}$$

$$\omega = \begin{bmatrix} \omega_1 \\ \omega_2 \\ \omega_3 \end{bmatrix}$$

$$K_4 = \frac{1}{2}\,\omega^T\, I\, \omega$$

Inertia tensor.

$$I = \begin{bmatrix} \int (y^2+z^2)\,dm & -\int xy\,dm & -\int xz\,dm \\ -\int xy\,dm & \int (x^2+z^2)\,dm & -\int yz\,dm \\ -\int xz\,dm & -\int yz\,dm & \int (x^2+y^2)\,dm \end{bmatrix}$$

K_4 is the rotational part of the kinetic energy.

$$\boxed{K = \frac{1}{2} m V_c^T V_c + \frac{1}{2}\,\omega^T I\, \omega}$$

Remarks:

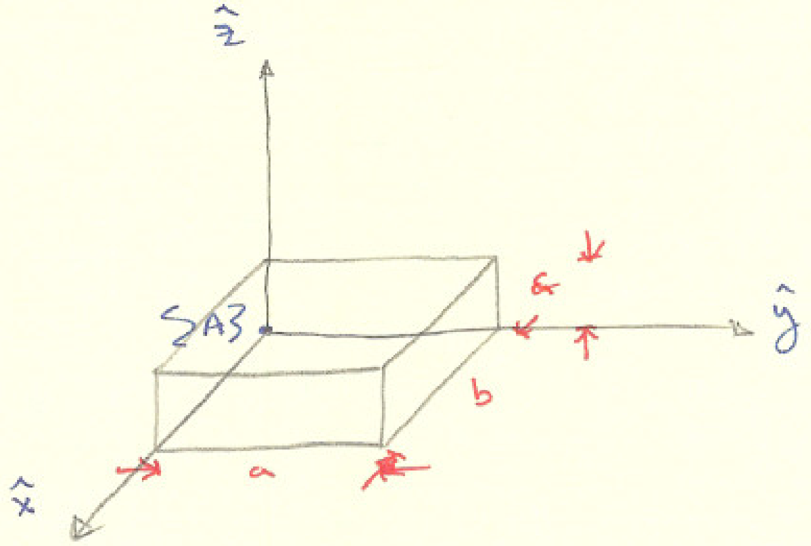
* The product $V_c^T V_c$ is the same irrispective to the frame in which the vectors V_c is expressed.

$$\bar{V}_c = R V_c$$

$$\bar{V}_c^T \bar{V}_c = \bar{V}_c^T R^T R V_c = V_c^T V_c$$

* For $\omega^T I \omega$; Since I is computed w.r.t the frame of the attached objcet, ω must be expressed w.r.t the same frame.

Ex:



Find the inertia matrix for the rectangular body of uniform density ρ w.r.t. frame $\{A\}$

$$I_{11} = \int_B (y^2+z^2)\,dm = \int_0^c\int_0^a\int_0^b (y^2+z^2)\rho\,dy\,dx\,dz$$

$$= \int_0^c\int_0^a (y^2+z^2)\rho\, b\,dy\,dz$$

$$= \int_0^c \left(\frac{a^3}{3} + z^3 a\right) b\rho\,dz$$

$$= \left(\frac{c a^3 b}{3} + \frac{c^3 a b}{3}\right)\rho$$

$$= \frac{m}{3}(a^2+c^2)$$

$$I_{22} = \frac{m}{3}(b^2+c^2)$$

$$I_{33} = \frac{m}{3}(a^2+b^2)$$

$$I_{12} = I_{21} = -\int_B xy\,dm$$

$$= -\int_0^c \int_0^a \int_0^b xy\,\rho\,dx\,dy\,dz$$

$$= -\frac{m}{4}ba$$

$$I_{31} = I_{13} = -\frac{m}{4}cb$$

$$I_{23} = I_{32} = -\frac{m}{4}ca$$

Inertia matrix is given (w.r.t. {A}) by:

$$I = \begin{bmatrix} \frac{m}{3}(a^2+c^2) & -\frac{m}{4}ba & -\frac{m}{4}cb \\ -\frac{m}{4}ba & \frac{m}{3}(b^2+c^2) & -\frac{m}{4}ca \\ -\frac{m}{4}cb & -\frac{m}{4}ca & \frac{m}{3}(a^2+b^2) \end{bmatrix}$$

The interia matrix of a retangular object w.r.t. a parrellel frame located at the center of mass is

$$I = \begin{bmatrix} \frac{m}{12}(a^2+c^2) & 0 & 0 \\ 0 & \frac{m}{12}(c^2+b^2) & 0 \\ 0 & 0 & \frac{m}{12}(a^2+b^2 \end{bmatrix}$$

Consider a manipulator consisting of n links.

$$V_{ci} = J_{vci}\,\dot{q}$$

translational part. for a point at the center of mass.

$$\omega_i = {}^0_iR^T\, J_{\omega i}\,\dot{q}$$

takes care of the expressing ω_i w.r.to the frame attached to link i.

angular part.

Note:
$$\begin{pmatrix} V_{ci} \\ \omega_i \end{pmatrix} = \begin{bmatrix} J_{vci} \\ J_{\omega i} \end{bmatrix} \dot{q}$$

Suppose that the mass of link i is m_i and the inertia matrix of link i (evaulated around a coordinate frame parallel to frame $\{i\}$ whose origin is at the center of mass at link i) is link I_i.

$$K = \frac{1}{2}\dot{q}^T \underbrace{\left[\sum_{i=1}^{n}\left(m_i J_{vci}^T J_{vci} + J_{\omega i}^T\, {}^0_iR\, I_i\, {}^0_iR^T J_{\omega i}\right)\right]}_{D(q)} \dot{q}$$

$$K = \frac{1}{2}\dot{q}^T D(q)\,\dot{q}$$

$D(q)$: mass inertia matrix of the manipulator

$D(q) \in \mathbb{R}^{n \times n}$

$D(q) = D^T(q)$

$D(q)$: possitive definate.